

# ON UNIFORM BOUNDEDNESS OF DYADIC AVERAGING OPERATORS IN SPACES OF HARDY-SOBOLEV TYPE

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**ABSTRACT.** We give an alternative proof of recent results by the authors on uniform boundedness of dyadic averaging operators in (quasi-)Banach spaces of Hardy-Sobolev and Triebel-Lizorkin type, in the largest possible range of parameters. This result served as the main tool to establish Schauder basis properties of suitable enumerations of the univariate Haar system in the mentioned spaces. The proof here is based on characterizations of the respective spaces in terms of compactly supported Daubechies wavelets.

## 1. INTRODUCTION

Consider the dyadic averaging operators  $\mathbb{E}_N$  on the real line given by

$$(1) \quad \mathbb{E}_N f(x) = \sum_{\mu \in \mathbb{Z}} \mathbb{1}_{I_{N,\mu}}(x) 2^N \int_{I_{N,\mu}} f(t) dt$$

with  $I_{N,\mu} = [2^{-N}\mu, 2^{-N}(\mu+1))$ .  $\mathbb{E}_N f$  is the conditional expectation of  $f$  with respect to the  $\sigma$ -algebra generated by the dyadic intervals of length  $2^{-N}$ . The following theorem on uniform boundedness in Triebel-Lizorkin spaces  $F_{p,q}^s$  was proved by the authors in [6] and serves as the main tool to establish that suitably regular enumerations of the Haar system form a Schauder basis for the spaces  $F_{p,q}^s$  in the parameter ranges of the theorem. Since the uniform boundedness result is interesting on its own we give an alternative proof based on wavelet theory to make it accessible for a broader readership.

**Theorem 1.1.** [6] *Let  $1/2 < p < \infty$ ,  $0 < q \leq \infty$ , and  $1/p - 1 < s < \min\{1/p, 1\}$ . Then there is a constant  $C := C(p, q, s) > 0$  such that for all  $f \in F_{p,q}^s$*

$$(2) \quad \sup_{N \in \mathbb{N}} \|\mathbb{E}_N f\|_{F_{p,q}^s} \leq C \|f\|_{F_{p,q}^s}.$$

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2010 *Mathematics Subject Classification.* 46E35, 46B15, 42C40.

*Key words and phrases.* Schauder basis, Unconditional bases, Haar system, Hardy-Sobolev space, Triebel-Lizorkin space.

G.G. was supported in part by grants MTM2013-40945-P and MTM2014-57838-C2-1-P from MINECO (Spain), and grant 19368/PI/14 from Fundación Séneca (Región de Murcia, Spain). A.S. was supported in part by NSF grant DMS 1500162. T.U. was supported the DFG Emmy-Noether program UL403/1-1.

In [6], this result served as the main tool to establish that suitably regular enumerations of the Haar system form a Schauder basis for the spaces  $F_{p,q}^s$  in the parameter ranges of the theorem, see §3. The connection with the Haar system is given via the martingale difference operators

$$\mathbb{D}_N = \mathbb{E}_{N+1} - \mathbb{E}_N$$

which are the orthogonal projections to the spaces generated by Haar functions with fixed Haar frequency  $2^N$ .

In previous works stronger notions of convergence have been examined, such as unconditional convergence for the martingale difference series. This is equivalent with the inequality

$$(3) \quad \left\| \sum_n b_n \mathbb{D}_n f \right\|_{F_{p,q}^s} \lesssim \|b\|_{\ell^\infty(\mathbb{N})} \|f\|_{F_{p,q}^s}.$$

It follows from the results in Triebel [14] that (3) holds if we add the condition  $1/q - 1 < s < 1/q$  to the hypotheses in the theorem. For the case  $q = 2$  this corresponds to the shaded region in Figure 1.

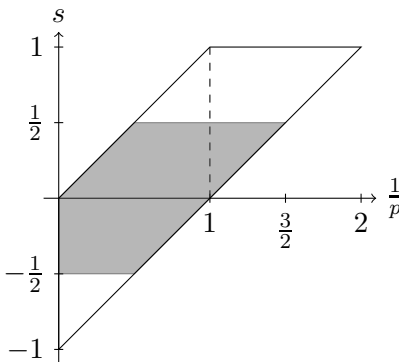


FIGURE 1. Unconditional convergence in Hardy-Sobolev spaces

It was shown in [10], [11] that the additional restriction on the  $q$ -parameter is necessary for (3) to hold. If we drop it then Theorem 1.1 and a summation by parts argument imply that (3) holds with the larger norm  $\|b\|_\infty + \|b\|_{BV}$ . It should be interesting to establish sharp results involving sequence spaces that are intermediate between  $\ell^\infty(\mathbb{N})$  and  $BV(\mathbb{N})$ . We remark that these problems are interesting only for the  $F_{p,q}^s$  spaces since inequality (3) with  $B_{p,q}^s$  in place of  $F_{p,q}^s$  holds in the full parameter range of Theorem 1.1, see [14] for further discussion and historical comments.

In §2 we give a proof of Theorem 1.1 using characterizations of Triebel-Lizorkin spaces based on Daubechies wavelets. In §3 we apply the methods to get an additional result needed to obtain the Schauder basis property of the Haar system.

## 2. PROOF OF THEOREM 1.1

We start with some preliminaries on convolution kernels which are used in Littlewood-Paley type decompositions. We use a characterization of the Triebel-Lizorkin spaces via Littlewood-Paley operators defined by dilates of compactly supported kernels, or so-called local means. Let  $\varphi_0, \varphi$  be Schwartz functions on the real line, compactly supported in  $(-1/2, 1/2)$  such that  $|\hat{\varphi}_0(\xi)| > 0$  on  $(-1/2, 1/2)$  and  $|\hat{\varphi}(\xi)| > 0$  on  $\{\xi \in \mathbb{R} : 1/8 < |\xi| < 1\}$ . Moreover  $\varphi$  has vanishing moments up to large order  $M$ , i.e.,

$$\int \varphi(x) x^n dx = 0 \quad \text{for } n = 0, 1, \dots, M.$$

Let  $\varphi_j := 2^j \varphi(2^j \cdot)$  and  $L_j f = \varphi_j * f$ . We then have

$$\|f\|_{F_{p,q}^s} \asymp \left\| \left( \sum_{j=0}^{\infty} 2^{jsq} |\varphi_j * f|^q \right)^{1/q} \right\|_p.$$

This result is closely related to the classical theorem of Benedek, Calderón and Panzone [1] on vector-valued singular integrals (at least for  $1 < p, q < \infty$ ). For the quasi-Banach case and further refinements we refer to Triebel's book [12, §2.4.6], Rychkov [9] and the references therein.

In addition to the characterization via local means we will use a characterization via compactly supported Daubechies wavelets [2], [15, Sect. 4]. Let  $\psi_0$  and  $\psi$  be the orthogonal scaling function and corresponding wavelet of Daubechies type such that  $\psi_0, \psi$  being sufficiently smooth ( $C^K$ ) and  $\psi$  having sufficiently many vanishing moments ( $L$ ). We denote

$$\psi_{j,\nu}(x) := \frac{1}{\sqrt{2}} \psi(2^{j-1}x - \nu) \quad , \quad j \in \mathbb{N}, \nu \in \mathbb{Z},$$

and  $\psi_{0,\nu}(x) := \psi_0(x - \nu)$  for  $\nu \in \mathbb{Z}$ . Let  $0 < p < \infty$ ,  $0 < q \leq \infty$  and  $s \in \mathbb{R}$ . If  $K$  and  $L$  are large enough (depending on  $p, q$  and  $s$ ) then we have the equivalent characterization,

$$(4) \quad \|f\|_{F_{p,q}^s} \asymp \left\| \left( \sum_{j=0}^{\infty} 2^{jsq} \left| \sum_{\nu \in \mathbb{Z}} \lambda_{j,\nu}(f) \mathbb{1}_{j,\nu} \right|^q \right)^{1/q} \right\|_p,$$

where  $\lambda_{j,\nu}(f) := 2^j \langle f, \psi_{j,\nu} \rangle$  and  $\mathbb{1}_{j,\nu}$  denotes the characteristic function of the interval  $I_{j,\nu} := [2^{-j}\nu, 2^{-j}(\nu + 1)]$ . See Triebel [13, Thm. 1.64] and the references therein. A corresponding characterization also holds true for Besov spaces  $B_{p,q}^s$ . Since we also deal with distributions which are not locally integrable, the inner product  $\langle f, \psi_{j,\nu} \rangle$  has to be interpreted in the usual way. Clearly,  $f$  can be decomposed into wavelet building blocks, i.e.

$$(5) \quad f = \sum_{j \in \mathbb{Z}} f_j \quad \text{with} \quad f_j = \begin{cases} \sum_{\nu \in \mathbb{Z}} \lambda_{j,\nu}(f) \psi_{j,\nu}, & \text{if } j \geq 0, \\ 0 & \text{if } j < 0. \end{cases}$$

Note, that the  $f_j$  represent  $K$  times continuously differentiable functions due to the regularity assumption on the wavelet.

**A. Proof in the case  $1/2 < p \leq 1$ .** Let  $1/p - 1 < s < 1$ . Let  $\{\varphi_j\}_{j \in \mathbb{N}}$  denote the local mean kernels from above. Using the decomposition (5) we can write with  $\theta := \min\{1, p, q\}$

$$\begin{aligned}
 \|\mathbb{E}_N f\|_{F_{p,q}^s} &\asymp \left\| \left( \sum_{j=0}^{\infty} |2^{js} \varphi_j * \mathbb{E}_N f|^q \right)^{1/q} \right\|_p \\
 &\lesssim \left( \sum_{\ell \in \mathbb{Z}} \left\| \left( \sum_{j=0}^{\infty} |2^{js} \varphi_j * \mathbb{E}_N f_{j+\ell}|^q \right)^{1/q} \right\|_p^\theta \right)^{1/\theta} \\
 (6) \quad &\lesssim \left( \sum_{\ell \in \mathbb{Z}} \left\| \left( \sum_{j+\ell \leq N} |2^{js} \varphi_j * \mathbb{E}_N f_{j+\ell}|^q \right)^{1/q} \right\|_p^\theta \right. \\
 (7) \quad &\quad \left. + \left\| \left( \sum_{j+\ell \geq N} |2^{js} \varphi_j * \mathbb{E}_N f_{j+\ell}|^q \right)^{1/q} \right\|_p^\theta \right)^{1/\theta}.
 \end{aligned}$$

We split the proof in several steps and distinguish several cases in the estimation of the  $p$ -norms in (6) and (7).

*Step A1.* Here we restrict to  $j + \ell \geq N$ . We deal with (7) and use that

$$(8) \quad \left\| \left( \sum_j |2^{js} \varphi_j * \mathbb{E}_N f_{j+\ell}|^q \right)^{1/q} \right\|_p^\theta \leq \sum_j \|2^{js} \varphi_j * \mathbb{E}_N f_{j+\ell}\|_p^\theta.$$

We continue estimating  $\|2^{js} \varphi_j * \mathbb{E}_N f_{j+\ell}\|_p$ . Note first that due to  $p \leq 1$

$$(9) \quad \|2^{js} \varphi_j * \mathbb{E}_N f_{j+\ell}\|_p \leq \left( \sum_{\nu \in \mathbb{Z}} |\lambda_{j+\ell, \nu}(f)|^p \|2^{js} \varphi_j * \mathbb{E}_N \psi_{j+\ell, \nu}\|_p^p \right)^{1/p}.$$

So it remains to deal with  $\|2^{js} \varphi_j * \mathbb{E}_N \psi_{j+\ell, \nu}\|_p$ . Note, that due to  $j + \ell \geq N$  the function  $\mathbb{E}_N \psi_{j+\ell, \nu}$  is a step function consisting of  $O(1)$  non-vanishing steps. These steps have length  $2^{-N}$  and magnitude bounded by  $O(2^{N-(j+\ell)})$ .

*Case A1.1* Assume  $j \geq N$ .

Due to the cancellation of  $\varphi_j$  and  $j \geq N$  we have that  $\varphi_j * \mathbb{E}_N \psi_{j+\ell, \nu}$  is supported on a union of intervals of total measure  $O(2^{-j})$  and bounded from above by  $O(2^{N-(j+\ell)})$ . This gives

$$(10) \quad \|2^{js} \varphi_j * \mathbb{E}_N \psi_{j+\ell, \nu}\|_p \lesssim 2^{js} 2^{-j/p} 2^{N-j-\ell}.$$

*Case A1.2.* Assume  $j \leq N$ .

Clearly, we have  $\ell \geq 0$  since  $j + \ell \geq N$ . Now  $\varphi_j * \mathbb{E}_N \psi_{j+\ell, \nu}$  is supported on an interval of size  $O(2^{-j})$ . As  $\mathbb{E}_N \psi_{j+\ell, \nu}$  consists of  $O(1)$  steps of length  $2^{-N}$  each and  $N \geq j$  we get by straightforward size estimates  $\varphi_j * \mathbb{E}_N \psi_{j+\ell, \nu} = O(2^{-\ell})$ . Hence

$$(11) \quad \|2^{js} \varphi_j * \mathbb{E}_N \psi_{j+\ell, \nu}\|_p \lesssim 2^{js} 2^{-j/p} 2^{N-j-\ell} 2^{j-N}.$$

*Step A2.* We consider  $j + \ell \leq N$  and use a different strategy to estimate (6).

*Case A2.1.* We first deal with the case  $j \leq N$  and estimate as follows

$$(12a) \quad \left\| \left( \sum_j |2^{js} \varphi_j * \mathbb{E}_N f_{j+\ell}|^q \right)^{1/q} \right\|_p^\theta \lesssim \left\| \left( \sum_j |2^{js} \varphi_j * [\mathbb{E}_N f_{j+\ell} - f_{j+\ell}]|^q \right)^{1/q} \right\|_p^\theta$$

$$(12b) \quad + \left\| \left( \sum_j |2^{js} \varphi_j * f_{j+\ell}|^q \right)^{1/q} \right\|_p^\theta.$$

Similar as in (8) we estimate the term in (12a) via

$$(13) \quad \left\| \left( \sum_j |2^{js} \varphi_j * [\mathbb{E}_N f_{j+\ell} - f_{j+\ell}]|^q \right)^{1/q} \right\|_p^\theta \leq \sum_j \|2^{js} \varphi_j * [\mathbb{E}_N f_{j+\ell} - f_{j+\ell}]\|_p^\theta.$$

Again, analogously to (9) we have

$$(14) \quad \|2^{js} \varphi_j * [\mathbb{E}_N f_{j+\ell} - f_{j+\ell}]\|_p \lesssim \left( \sum_{\nu \in \mathbb{Z}} |\lambda_{j+\ell, \nu}(f)|^p \|2^{js} \varphi_j * [\mathbb{E}_N \psi_{j+\ell, \nu} - \psi_{j+\ell, \nu}]\|_p^p \right)^{1/p}.$$

Using the mean value theorem together with (1) we see for all  $x \in \mathbb{R}$  that

$$|\mathbb{E}_N \psi_{j+\ell, \nu}(x) - \psi_{j+\ell, \nu}(x)| \leq 2^{j+\ell-N}.$$

Due to  $j + \ell \leq N$ , its support has length  $O(2^{-(j+\ell)})$  around  $\nu 2^{-(j+\ell)}$ . We continue distinguishing the cases  $\ell \geq 0$  and  $\ell < 0$ .

*Case A2.1.1.* Let  $\ell \geq 0$ . Since  $j + \ell \geq j$  the convolution with  $\varphi_j$  gives an additional factor  $2^{-\ell}$  and increases the support to an interval of size  $O(2^{-j})$ . Hence, we get

$$(15) \quad \|2^{js} \varphi_j * [\mathbb{E}_N \psi_{j+\ell, \nu} - \psi_{j+\ell, \nu}]\|_p \lesssim 2^{js} 2^{j+\ell-N} 2^{-\ell} 2^{-j/p}.$$

*Case A2.1.2.* Assume  $\ell \leq 0$ . This time the convolution with  $\varphi_j$  does not give an extra factor and the support of  $\varphi_j * [\mathbb{E}_N \psi_{j+\ell, \nu} - \psi_{j+\ell, \nu}]$  has length  $2^{-(j+\ell)}$ . Thus, we have in this case

$$(16) \quad \|2^{js} \varphi_j * [\mathbb{E}_N \psi_{j+\ell, \nu} - \psi_{j+\ell, \nu}]\|_p \lesssim 2^{js} 2^{j+\ell-N} 2^{-(j+\ell)/p}.$$

It remains to deal with the term in (12b). Since  $\psi_{j+\ell, \nu}$  and  $\varphi_j$  are sufficiently smooth and have sufficiently many vanishing moments well-known convolution inequalities, see for instance [7, p. 466] for the most general version or Frazier, Jawerth [4, Lem. 3.3], [5, Lem. B.1, B.2], yield

$$(17) \quad |\varphi_j * f_{j+\ell}(x)| \lesssim 2^{-|\ell|M} \sum_{\nu \in \mathbb{Z}} |\lambda_{j+\ell, \nu}(f)| (1 + 2^{\min\{j, j+\ell\}} |x - \nu 2^{-(j+\ell)}|)^{-R},$$

where  $M$  depends on the number of vanishing moments of the wavelet  $\psi$  and  $R$  can be chosen arbitrary large due to the compact support of  $\psi_{j+\ell}$  and  $\varphi$ . Next we apply [8, Lem. 7.1] to (17) which yields that for any  $0 < r < 1$  and  $R > 1/r$

$$(18) \quad \begin{aligned} & \sum_{\nu \in \mathbb{Z}} |\lambda_{j+\ell, \nu}(f)| (1 + 2^{\min\{j, j+\ell\}} |x - \nu 2^{-(j+\ell)}|)^{-R} \\ & \lesssim 2^{\ell+/r} \left( M_{\text{HL}} \left[ \left| \sum_{\nu \in \mathbb{Z}} \lambda_{j+\ell, \nu}(f) \mathbb{1}_{j+\ell, \nu} \right|^r \right] (x) \right)^{1/r}, \quad x \in \mathbb{R}. \end{aligned}$$

If the order of the Daubechies wavelet system (resulting in smoothness and vanishing moments) is now chosen such that  $M$  in (17) is larger than  $1/r + 1$  there is a positive  $\delta > 0$  such that

$$(19) \quad \begin{aligned} & \left\| \left( \sum_j |2^{js} \varphi_j * f_{j+\ell}|^q \right)^{1/q} \right\|_p \\ & \lesssim 2^{-|\ell|\delta} \left\| \left( \sum_j \left( M_{\text{HL}} \left[ \left| 2^{(j+\ell)s} \sum_{\nu \in \mathbb{Z}} \lambda_{j+\ell, \nu}(f) \mathbb{1}_{j+\ell, \nu} \right|^r \right] \right)^{q/r} \right)^{1/q} \right\|_p. \end{aligned}$$

Choosing  $r < \min\{p, q, 1\}$  we can apply Fefferman-Stein maximal inequality [3] which, together with (4), yields

$$(20) \quad \left\| \left( \sum_j |2^{js} \varphi_j * f_{j+\ell}|^q \right)^{1/q} \right\|_p \lesssim 2^{-|\ell|\delta} \|f\|_{F_{p,q}^s}.$$

*Case A2.2.* Assume  $j \geq N \geq j + \ell$  which implies  $\ell \leq 0$ . Using (8) and (9) again we reduce everything to estimating  $\|2^{js} \varphi_j * \mathbb{E}_N \psi_{j+\ell, \nu}\|_p$ . Due to the step function  $\mathbb{E}_N \psi_{j+\ell, \nu}$  and the cancellation of the  $\varphi_j$  we have

$$(21) \quad \begin{aligned} & 2^{js} \|\varphi_j * \mathbb{E}_N \psi_{j+\ell, \nu}\|_p \\ & \lesssim 2^{js} \left( \sum_{\mu \in \mathbb{Z}} \int_{|x-2^{-N}\mu| \lesssim 2^{-j}} |\varphi_j * \mathbb{E}_N \psi_{j+\ell, \nu}(x)|^p dx \right)^{1/p} \\ & \lesssim 2^{js} \left( \sum_{\mu \in \mathbb{Z}} \int_{|x-2^{-N}\mu| \lesssim 2^{-j}} |\varphi_j * [\mathbb{E}_N \psi_{j+\ell, \nu} - \psi_{j+\ell, \nu}](x)|^p dx \right)^{1/p} \\ & \quad + 2^{js} \left( \sum_{\mu \in \mathbb{Z}} \int_{|x-2^{-N}\mu| \lesssim 2^{-j}} |\varphi_j * \psi_{j+\ell, \nu}(x)|^p dx \right)^{1/p} \\ & \lesssim 2^{js} 2^{j+\ell-N} 2^{[N-(j+\ell)-j]/p} + \|2^{js} \varphi_j * \psi_{j+\ell, \nu}\|_p, \end{aligned}$$

where we took into account that the  $\mu$ -sum consists of  $O(2^{N-(j+\ell)})$  summands. It remains to deal with the quantity  $\|2^{js} \psi_{j+\ell, \nu} * \varphi_j\|_p$  in (21). By the same convolution inequality as used in (17) we obtain  $\|2^{js} \psi_{j+\ell, \nu} * \varphi_j\|_p \lesssim 2^{-|\ell|\delta}$  if the wavelet system has enough smoothness and vanishing moments.

Hence,

$$(22) \quad 2^{js} \|\varphi_j * \mathbb{E}_N \psi_{j+\ell, \nu}\|_p \lesssim 2^{js} 2^{j+\ell-N} 2^{[N-(j+\ell)-j]/p} + 2^{-\delta|\ell|}.$$

*Step A3. Estimation of (7).* Plugging (8), (9) and (10) into (7) yields

$$(23) \quad \left( \sum_{\ell \in \mathbb{Z}} \left\| \left( \sum_{j \geq \max\{N-\ell, N\}}^{\infty} |2^{js} \varphi_j * \mathbb{E}_N f_{j+\ell}|^q \right)^{1/q} \right\|_p^\theta \right)^{1/\theta} \\ \lesssim A_N \sup_{j, \ell} \left( \sum_{\nu \in \mathbb{Z}} |2^{(j+\ell)s} \lambda_{j+\ell, \nu}(f)|^p 2^{-(j+\ell)} \right)^{1/p}.$$

with

$$A_N^\theta = \sum_{j \geq N} 2^{(N-j)\theta} \sum_{\ell \geq N-j} 2^{\theta\ell(1/p-1-s)} \lesssim 1,$$

uniformly in  $N$ , by the assumption  $s > 1/p - 1$ . Furthermore,

$$(24) \quad \sup_{j, \ell} \left( \sum_{\nu \in \mathbb{Z}} |2^{(j+\ell)s} \lambda_{j+\ell, \nu}(f)|^p 2^{-(j+\ell)} \right)^{1/p} = \sup_j \left\| 2^{js} \sum_{\nu \in \mathbb{Z}} \lambda_{j, \nu}(f) \mathbb{1}_{j, \nu} \right\|_p \\ \leq \left\| \sup_j 2^{js} \left| \sum_{\nu \in \mathbb{Z}} \lambda_{j, \nu}(f) \mathbb{1}_{j, \nu} \right| \right\|_p \lesssim \|f\|_{F_{p, q}^s},$$

where we used (4) in the last estimate.

Plugging (8), (9) and (11) into (7) leads to a similar estimate as above, only the sums over  $j$  and  $\ell$  change to

$$\tilde{A}_N^\theta = \sum_{j \leq N} \sum_{\ell \geq N-j} 2^{\theta\ell(1/p-1-s)}$$

which is uniformly bounded in  $N$  if  $s > 1/p - 1$ .

*Step A4. Estimation of (6).* Replace (6) by (12a) and (12b) and observe that (20) and summing over  $\ell$  already yields the desired bound for (12b). It remains to deal with (12a). Combining (13), (14), (15) and (24) we find

$$(25) \quad \sum_{\ell=-\infty}^N \left\| \left( \sum_{j=0}^{\min\{N-\ell, N\}} |2^{js} \varphi_j * [\mathbb{E}_N f_{j+\ell} - f_{j+\ell}]|^q \right)^{1/q} \right\|_p^\theta \\ \lesssim \left( \sum_{j \leq N} 2^{(j-N)\theta} \sum_{\ell=-\infty}^{N-j} 2^{\theta\ell(1/p-s)} \right)^{1/\theta} \|f\|_{F_{p, q}^s}.$$

The sums are finite and uniformly bounded if  $1/p - 1 < s < 1/p$ . Together with (24) we obtain the desired bound in case  $\ell \geq 0$  (see Case A2.1.1).

Combining (13), (14) and (16) leads to a similar calculation where the sums over  $j$  and  $\ell$  change to

$$\sum_{j \leq N} 2^{(j-N)\theta} \sum_{\ell \leq 0} 2^{\theta\ell(1-s)},$$

which is uniformly bounded if  $s < 1$ .

Finally, we combine (8), (9), (22) and (24) to obtain

$$\begin{aligned}
(26) \quad & \left( \sum_{\ell \leq 0} \left\| \left( \sum_{j=N}^{N-\ell} |2^{js} \varphi_j * \mathbb{E}_N f_{j+\ell}|^q \right)^{1/q} \right\|^\theta \right)^{1/\theta} \\
& \lesssim \left( \sum_{j \geq N} 2^{(j-N)\theta} 2^{\theta(N-j)/p} \sum_{\ell=-\infty}^{N-j} 2^{\theta\ell(1-s)} + \sum_{j \geq N} \sum_{\ell=-\infty}^{N-j} 2^{-\theta\delta|\ell|} \right)^{1/\theta} \|f\|_{F_{p,q}^s},
\end{aligned}$$

which is uniformly bounded if  $s < 1$ . Combine (25) and (26) to complete the estimation of (12a). This concludes the proof in the case  $p \leq 1$ .  $\square$

**B. Proof in the case  $1 < p < \infty$ .** We follow the proof in the case  $p \leq 1$  until (8) and (13), respectively. Then we have to proceed differently.

*Case B1.1* Assume  $N \leq j, j + \ell$ . We replace (9) by

$$\begin{aligned}
(27) \quad & \|2^{js} \varphi_j * \mathbb{E}_N f_{j+\ell}\|_p^p \\
& \leq \int \left[ \sum_{\nu \in \mathbb{Z}} |2^{js} \lambda_{j+\ell,\nu}(f)| \cdot |\varphi_j * \mathbb{E}_N \psi_{j+\ell,\nu}(x)| \right]^p dx \\
& \lesssim \sum_{\nu \in \mathbb{Z}} |2^{js} \lambda_{j+\ell,\nu}(f)|^p 2^{-j} 2^{(N-j-\ell)p}.
\end{aligned}$$

Indeed, since  $\mathbb{E}_N \psi_{j+\ell,\nu} = 0$  if  $\text{supp } \psi_{j+\ell,\nu} \subset I_{N,\mu}$  the sum on the right-hand side of (27) is lacunary and the functions  $\varphi_j * \mathbb{E}_N \psi_{j+\ell,\nu}$  have essentially disjoint support. Hence, we get

$$\begin{aligned}
(28) \quad & \|2^{js} \varphi_j * \mathbb{E}_N f_{j+\ell}\|_p \\
& \lesssim 2^{-\ell s} 2^{N-j-\ell} 2^{\ell/p} \left( \sum_{\nu \in \mathbb{Z}} |2^{(j+\ell)s} \lambda_{j+\ell,\nu}(f)|^p 2^{-(j+\ell)} \right)^{1/p}.
\end{aligned}$$

For  $1/p - 1 < s < 1/p$  the sum over the respective range of  $j$  and  $\ell$  is uniformly bounded.

*Case B1.2.* We now deal with  $j + \ell \geq N \geq j$ . Here we estimate

$$(29) \quad \|2^{js} \varphi_j * \mathbb{E}_N f_{j+\ell}\|_p \lesssim \|2^{js} \varphi_j * (\mathbb{E}_N f_{j+\ell} - f_{j+\ell})\|_p + \|2^{js} \varphi_j * f_{j+\ell}\|_p.$$

The second summand is estimated using (17) and (18). This results in

$$(30) \quad \|2^{js} \varphi_j * f_{j+\ell}\|_p \lesssim 2^{-\delta\ell} \left( \sum_{\nu} |2^{(j+\ell)s} \lambda_{j+\ell,\nu}(f)|^p 2^{-(j+\ell)} \right)^{1/p}.$$

To estimate the first summand on the right-hand side of (29) we are going to exploit the cancellation property

$$(31) \quad \mathbb{E}_N(f - \mathbb{E}_N f) = 0.$$



We continue estimating the first summand on the right-hand side of (29). Using (31) we write

(32)

$$\begin{aligned}
& \left| 2^{js} \int \varphi_j(x-y) (\mathbb{E}_N f_{j+\ell}(y) - f_{j+\ell}(y)) dy \right| \\
&= \left| 2^{js} \sum_{\mu: |2^{-N}\mu-x| \lesssim 2^{-j}} \int_{I_{N,\mu}} \varphi_j(x-y) (\mathbb{E}_N f_{j+\ell}(y) - f_{j+\ell}(y)) dy \right| \\
&= \left| 2^{js} \sum_{\mu: |2^{-N}\mu-x| \lesssim 2^{-j}} \int_{I_{N,\mu}} (\varphi_j(x-y) - \varphi_j(x-2^{-N}\mu)) (\mathbb{E}_N f_{j+\ell}(y) - f_{j+\ell}(y)) dy \right|
\end{aligned}$$

and continue to estimate this by

$$\begin{aligned}
& 2^{js} \sum_{\mu: |2^{-N}\mu-x| \lesssim 2^{-j}} \int_{I_{N,\mu}} |(\varphi_j(x-y) - \varphi_j(x-2^{-N}\mu)) \cdot \mathbb{E}_N f_{j+\ell}(y)| dy \\
&+ \left| 2^{js} \sum_{\mu: |2^{-N}\mu-x| \lesssim 2^{-j}} \int_{I_{N,\mu}} (\varphi_j(x-y) - \varphi_j(x-2^{-N}\mu)) \cdot f_{j+\ell}^{\mu,1}(y) dy \right| \\
&+ \left| 2^{js} \sum_{\mu: |2^{-N}\mu-x| \lesssim 2^{-j}} \int_{I_{N,\mu}} (\varphi_j(x-y) - \varphi_j(x-2^{-N}\mu)) \cdot f_{j+\ell}^{\mu,2}(y) dy \right| \\
&=: F_0(x) + F_1(x) + F_2(x),
\end{aligned}$$

where

$$\begin{aligned}
f_{j+\ell}^\mu &:= \sum_{\nu: \text{supp } \psi_{j+\ell,\nu} \cap I_{N,\mu} \neq \emptyset} \lambda_{j+\ell,\nu}(f) \psi_{j+\ell,\nu}, \\
f_{j+\ell}^{\mu,1} &:= \sum_{\nu: \text{supp } \psi_{j+\ell,\nu} \subset I_{N,\mu}} \lambda_{j+\ell,\nu}(f) \psi_{j+\ell,\nu}, \\
f_{j+\ell}^{\mu,2} &:= f_{j+\ell}^\mu - f_{j+\ell}^{\mu,1}.
\end{aligned}$$

$F_0(x)$  can be estimated by

$$2^{js} \sum_{\mu: |2^{-N}\mu-x| \lesssim 2^{-j}} 2^{2j-2N} \sup_{y \in I_{N,\mu}} \sum_{\nu: \text{supp } \psi_{j+\ell,\nu} \cap I_{N,\mu} \neq \emptyset} |\lambda_{j+\ell,\nu}(f) \mathbb{E}_N(\psi_{j+\ell,\nu})(y)|.$$

Here  $\mathbb{E}_N \psi_{j+\ell,\nu}$  is mostly vanishing, namely when  $\text{supp } \psi_{j+\ell,\nu} \subset I_{N,\mu}$ . If it does not vanish then the boundary of  $I_{N,\mu}$  intersects  $\text{supp } \psi_{j+\ell,\nu}$  and  $|\mathbb{E}_N \psi_{j+\ell,\nu}| \lesssim 2^{N-(j+\ell)}$ . This happens only for a bounded number of  $\nu$ 's (independently of  $j, \ell$ ). Thus for a fixed  $y$  only a bounded number of coefficients contribute. Hence, we have

(33)

$$F_0(x) \lesssim 2^{js} 2^{2j-2N} 2^{N-(j+\ell)} \sum_{\mu: |2^{-N}\mu-x| \lesssim 2^{-j}} \sup_{\nu: \text{supp } \psi_{j+\ell,\nu} \cap \partial I_{N,\mu} \neq \emptyset} |\lambda_{j+\ell,\nu}(f)|.$$

Taking the  $L_p$ -norm and using Hölder's inequality with  $1/p + 1/p' = 1$  yields

$$(34) \quad \|F_0\|_p \lesssim 2^{-\ell s} 2^{2j-2N} 2^{N-(j+\ell)} 2^{(N-j)/p'} 2^{\ell/p} \times \left( \sum_{\nu} |2^{(j+\ell)s} \lambda_{j+\ell,\nu}(f)|^p 2^{-(j+\ell)} \right)^{1/p}.$$

To estimate  $F_1$  we observe, setting  $g_{j+\ell}^1 := \sum_{\mu} f_{j+\ell}^{\mu,1}$ ,

$$F_1(x) = \left| 2^{js} \int \varphi_j(x-y) \left( \sum_{\mu: |2^{-N}\mu-x| \lesssim 2^{-j}} f_{j+\ell}^{\mu,1}(y) \right) dy \right| = |2^{js} \varphi_j * g_{j+\ell}^1(x)|.$$

With a similar reasoning as in (30) and a monotonicity argument we achieve

$$\|F_1\|_p \lesssim 2^{-\delta\ell} \left( 2^{(j+\ell)s} \sum_{\nu \in \mathbb{Z}} |\lambda_{j+\ell,\nu}(f)|^p 2^{-(j+\ell)} \right)^{1/p}.$$

Finally, we deal with  $F_2(x)$ . Since to  $f_{j+\ell}^{\mu,2}$  only a uniformly bounded number of coefficients  $\lambda_{j+\ell,\nu}$  contribute to the sum and the integrals are taken over an interval of length  $O(2^{-(j+\ell)})$  we obtain, similar as above, by Hölder's inequality

$$(35) \quad \|F_2\|_p \lesssim 2^{-\ell s} 2^{-\ell+j-N} 2^{(N-j)/p'} 2^{\ell/p} \left( \sum_{\nu} |2^{(j+\ell)s} \lambda_{j+\ell,\nu}(f)|^p 2^{-(j+\ell)} \right)^{1/p}.$$

Putting the estimates from (29) to (35) together we observe that the sum over the respective range of  $j$  and  $\ell$  (see (7)) is uniformly bounded with respect to  $N$  if  $s > 1/p - 1$ .

*Case B2.1.* Here we deal with  $j + \ell, j \leq N$ . We return to (12a) and estimate the first summand as done in (13). We continue similarly as after (31) and obtain the pointwise estimate (32). Since  $j + \ell \leq N$  there is only a bounded number of coefficients  $\lambda_{j+\ell,\nu}(f)$  contributing to  $f_{j+\ell}$  on  $I_{N,\mu}$ . Using the mean value theorem in both factors of the integral in (32) we obtain

$$|2^{js} [\mathbb{E}_N(f_{j+\ell}) - f_{j+\ell}] * \varphi_j(x)| \lesssim 2^{js} 2^{2j-2N} 2^{j+\ell-N} \times \sum_{\mu: |2^{-N}\mu-x| \lesssim 2^{-j}} \sup_{|\nu 2^{-(j+\ell)} - 2^{-N}\mu| \lesssim 1} |\lambda_{j+\ell,\nu}(f)|,$$

which yields

$$\begin{aligned} & \|2^{js} [\mathbb{E}_N(f_{j+\ell}) - f_{j+\ell}] * \varphi_j\|_p \\ & \lesssim 2^{-\ell s} 2^{j+\ell-N} 2^{2j-2N} 2^{(N-j)/p'} 2^{\ell/p} \left( \sum_{\nu \in \mathbb{Z}} |2^{(j+\ell)s} \lambda_{j+\ell,\nu}(f)|^p 2^{-(j+\ell)} \right)^{1/p}. \end{aligned}$$

The sum over the respective  $j$  and  $\ell$  is uniformly bounded in  $N$  whenever  $-1 < s < 1 + 1/p$ . To estimate the second term on the right-hand side of (12a) we literally follow the arguments in (17) and below to end up with (20).

*Case B2.2.* Finally  $j + \ell \leq N \leq j$ . Instead of (21) we estimate as follows.

$$\begin{aligned}
 & \|2^{js}\varphi_j * \mathbb{E}_N f_{j+\ell}\|_p \\
 & \leq 2^{js} \left( \sum_{\mu \in \mathbb{Z}} \int_{|x-2^{-N}\mu| \lesssim 2^{-j}} |\varphi_j * [\mathbb{E}_N f_{j+\ell} - f_{j+\ell}](x)|^p dx \right)^{1/p} \\
 & + 2^{js} \left( \sum_{\mu \in \mathbb{Z}} \int_{|x-2^{-N}\mu| \lesssim 2^{-j}} |\varphi_j * f_{j+\ell}(x)|^p dx \right)^{1/p}.
 \end{aligned} \tag{36}$$

The second summand on the right-hand side can be estimated by

$$2^{-|\ell|\delta} \left( \sum_{\nu \in \mathbb{Z}} |2^{(j+\ell)s} \lambda_{j+\ell,\nu}(f)|^p 2^{-(j+\ell)} \right)^{1/p}, \tag{37}$$

whereas, similar to (21), the first summand in (36) is bounded by

$$2^{-\ell s} 2^{j+\ell-N} 2^{(N-j)/p} \left( \sum_{\nu \in \mathbb{Z}} |2^{(j+\ell)s} \lambda_{j+\ell,\nu}(f)|^p 2^{-(j+\ell)} \right)^{1/p}. \tag{38}$$

Altogether we encounter the condition  $1/p - 1 < s < 1/p$  for any  $0 < q \leq \infty$  for the uniform boundedness of  $\mathbb{E}_N : F_{p,q}^s \rightarrow F_{p,q}^s$  in case  $1 \leq p < \infty$ .  $\square$

### 3. ON THE SCHAUDER BASIS PROPERTY FOR THE HAAR SYSTEM.

Let  $\{h_{N,\mu} : \mu \in \mathbb{Z}\}$  be the set of Haar functions with Haar frequency  $2^{-N}$  and define for  $N \in \mathbb{N}_0$  and sequences  $a \in \ell^\infty(\mathbb{Z})$ ,

$$T_N[f, a] = \sum_{\mu \in \mathbb{Z}} a_\mu 2^N \langle f, h_{N,\mu} \rangle h_{N,\mu}. \tag{39}$$

In particular for the choice of  $a = (1, 1, 1, \dots)$  one recovers the operator  $\mathbb{E}_{N+1} - \mathbb{E}_N$ . It was shown in [6] that Theorem 1.1 together with

$$\sup_{N \in \mathbb{N}} \sup_{\|a\|_\infty \leq 1} \|T_N[f, a]\|_{B_{p,q}^s} \leq C \|f\|_{B_{p,\infty}^s}, \tag{40}$$

$$1/2 < p \leq \infty, 0 < q \leq \infty, \text{ and } 1/p - 1 < s < \min\{1/p, 1\},$$

implies Schauder basis properties for suitable enumerations of the Haar system. For the sake of completeness we give a sketch of this inequality which relies on the arguments in the previous section.

*Proof of (40).* We may assume  $\|a\|_\infty = 1$ . The modification of the proof of Theorem 1.1 is the fact that, due to the cancellation properties of the Haar functions participating in (39), we do not need the splittings in (12a), (21), (29), and (36) and the subsequent considerations like (17) – (20). Therefore, we may start with a Besov norm  $\|\cdot\|_{B_{p,q}^s}$  on the left-hand side (see (6), (7), (8)) and always end up with the Besov norm  $\|\cdot\|_{B_{p,\infty}^s}$  on the right-hand side, see (23), (28), (34) and the comments below.

*Case 1.1.* Suppose  $j + \ell, j \geq N$ . The estimates in (27), (28) apply almost literally to  $\|2^{js}\varphi_j * T_N[\psi_{j+\ell,\nu}, a]\|_p$  and yields estimates which are uniform

for  $\|a\|_\infty = 1$ . Note, that we did not yet need any cancellation of the Haar functions.

*Case 1.2.* Suppose  $j + \ell \geq N \geq j$ . The splitting in (29) is not necessary anymore, we can work directly with  $\|2^{js}T_N[f_{j+\ell}, a]\|_p$ . An analogous identity to (32) holds true with  $\mathbb{E}_N(f_{j+\ell}) - f_{j+\ell}$  replaced by  $T_N[f_{j+\ell}, a]$  due to the cancellation of the Haar functions  $h_{N,\mu}$ . In what follows we only have to care for a counterpart of  $F_0$  since  $F_1$  and  $F_2$  do not show up. We end up with a counterpart of (34) for  $\|2^{js}T_N[f_{j+\ell}, a]\|_p$ .

*Case 2.1.* Suppose  $N \geq j + \ell, j$ . Again, due to the cancellation of the Haar function, a splitting as in (12a) is not necessary and we obtain a version of (32) as in Case 1.2. The mean value theorem applied to the first factor in the integral gives the factor  $2^{2j-2N}$ , whereas the cancellation of  $h_{N,\mu}$  gives  $|T_N(\psi_{j+\ell,\nu})(x)| \lesssim 2^{j+\ell-N}$ . We continue as in the proof of Theorem 1.1.

*Case 2.2.* The remaining case  $j + \ell \leq N \leq j$  goes analogously to Case B2.2. in the proof of Theorem 1.1. Note, that also here the splitting in (36) and the subsequent consideration for the second summand on the right-hand side is not necessary.  $\square$

**Acknowledgment.** The authors worked on this project while participating in the 2016 summer program in Constructive Approximation and Harmonic Analysis at the Centre de Recerca Matemàtica. They would like to thank the organizers of the program for providing a pleasant and fruitful research atmosphere.

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